THE OPTICAL HELICITY IN A MORE ALGEBRAIC APPROACH TO ELECTROMAGNETISM

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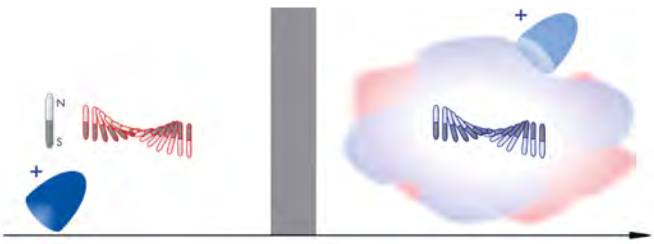

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The constant miniaturization trend in nanophotonics challenges some of the theoretical tools at our disposal: Analytical shortcuts such as the dipolar and paraxial approximation become unapplicable, and the computational cost of fully numerical studies often renders them impractical. A basic electromagnetic property, the optical helicity, and the use of more abstract tools inspired by it, are rising to meet the challenge.

The optical helicity, also known as electromagnetic helicity, represents the handedness of Maxwell fields, and is the conserved quantity connected to the electromagnetic duality symmetry. This connection allows to consider the polarization of the field within the powerful framework of symmetries and conservation laws. Such framework is formalized in electromagnetic Hilbert spaces thanks to the existence of an appropriate scalar product for Maxwell fields [1]. While these ideas may seem of purely theoretical interest, the abstraction and generality that they afford facilitate both the understanding and the design of specific light-matter interaction effects. The algebraic treatment of helicity and its eigenstates [2] is also a valuable tool for extending the theory, as exemplified by the recently established new connection between optics and magnetism [3].

FUNDAMENTALS AND APPLICATIONS

Helicity is the projection of the angular momentum vector **J** onto the direction of the linear momentum vector **P**,

$$\Lambda = \frac{\mathbf{J} \cdot \mathbf{P}}{|\mathbf{P}|}. \qquad (1)$$

Equation 1 is the most general form of the helicity operator, which is valid for many particles and fields, such as electrons and gravitational waves. For Maxwell fields, helicity describes the sense of screw in light: The circular polarization handedness.

Helicity is a pseudoscalar: Its sign flips under any spatial inversion operation such as parity or mirror reflections. Figure 1 illustrates some transformation properties of helicity comparing them with the transformation properties of angular momentum. Helicity and angular momentum are two different properties of the field. A very basic difference is that, while a two dimensional plane is enough to define a rotation (angular momentum), three spatial dimensions are required to define a sense of screw (helicity). The systematic consideration of how helicity and angular momentum transform allows the identification of the symmetry reasons for particular light-matter interaction effects, which can then be used for the analysis and prediction of experimental measurements. For example, the optical vortices seen in focusing by or scattering off cylindrically symmetric systems are readily shown to be due to the breaking of a different

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symmetry in each case (see Chap. 3 [4]): Translational symmetry in focusing, and electromagnetic duality symmetry (see below) in scattering. For the directional coupling of emitters onto waveguides, it is readily shown that the polarization handedness of the emission cannot be responsible for the directionality, which is controlled by angular momentum [5], see Fig. 3.

For transverse Maxwell fields, helicity has two possible eigenvalues, $\lambda = +1$ and $\lambda = -1$, whose corresponding eigenstates for $(\mathbf{r},t)$ -dependent Maxwell fields are (SI units assumed):

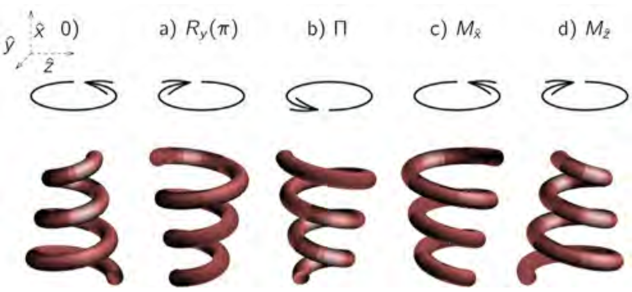
$$\mathbf{F}_\lambda(\mathbf{r},t) = \sqrt{\frac{\epsilon_0}{2}}[\mathbf{E}(\mathbf{r},t) + i\lambda c_0 \mathbf{B}(\mathbf{r},t)],\ \Lambda\mathbf{F}_\lambda(\mathbf{r},t) = \lambda\mathbf{F}_\lambda(\mathbf{r},t). \qquad (2)$$

The $\mathbf{F}_\lambda(\mathbf{r},t)$ can be built as the following sum of plane waves:

$$\mathbf{F}_\lambda(\mathbf{r},t) = \int_{\mathbb{R}^3-\{0\}} \frac{d\mathbf{k}}{\sqrt{(2\pi)^3}} \mathbf{F}_\lambda(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r} - ic_0|\mathbf{k}|t), \qquad (3)$$

where $\mathbf{k}\cdot\mathbf{F}_\lambda(\mathbf{k}) = 0$, $i\hat{\mathbf{k}} \times \mathbf{F}_\lambda(\mathbf{k}) = \lambda\mathbf{F}_\lambda(\mathbf{k})$ since $\Lambda \equiv i\hat{\mathbf{k}}\times$ in this representation and, importantly, the frequency is restricted to positive values $\omega = c_0|\mathbf{k}| > 0$. The $\mathbf{F}_\lambda(\mathbf{r}, t)$ are the positive frequency restriction of the Riemann-Silberstein vectors • • •

Figure 1. Transformations of helicity, represented by the helices, and of angular momentum, represented by the 2D turns. The initial objects in a) are transformed by a rotation by 180 degrees along the y axis in b), by the parity operation in c), by a mirror reflection across the YZ plane in d), and by a mirror reflection across the XY plane in e). Spatial inversion transformations always flip the screw sense of the helix, while rotations never do. The chosen angular momentum changes sign (the sense of the turn is inverted) upon the rotation in b) and the reflection in e), and stays invariant upon parity and the reflection in d).



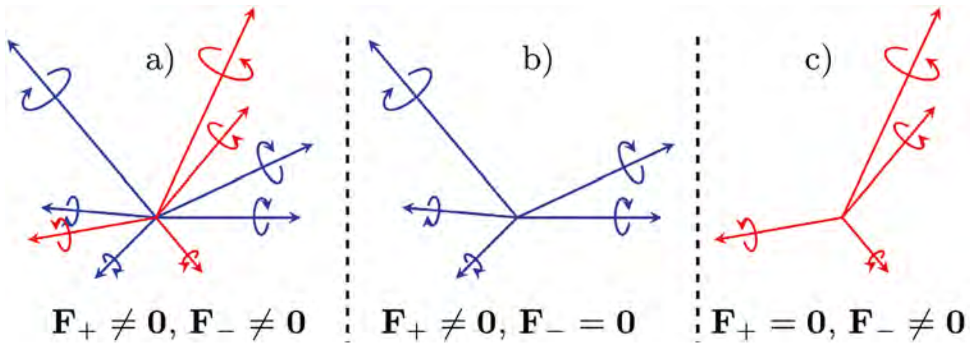


Figure 2. $F_\pm$ split a general electromagnetic field such as the arbitrary multi-frequency combination of circularly polarized plane waves in a) into its two helicity components, left-handed (blue) in b), and right-handed (red) in c).

[2], and its monochromatic components are also known as Beltrami fields [6]. As illustrated in Fig. 2, the $\mathbf{F}_\lambda(\mathbf{r}, t)$ split the electromagnetic field into its left and right circular polarization handedness, for $\lambda = +1$ and $\lambda = -1$, respectively. The restriction to positive frequencies makes $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ necessarily complex valued, and is crucial for achieving the handedness splitting: If $\mathbf{E}(\mathbf{r}\ t)$and $\mathbf{B}(\mathbf{r}, t)$ are real-valued, then $|\mathbf{F}_+(\mathbf{r}, t)| = |\mathbf{F}_-(\mathbf{r}, t)|$follows from Eq. 2, negating the handedness separation.

The splitting allows to analyze the handedness of near fields around illuminated nanostructures, such as the silicon disk in Fig.4, which is illuminated on axis by a single plane wave of positive helicity. Figures 4a) and 4b) correspond to illumination with two different frequencies. Each sub-figure is divided into two areas where the false color scale shows point-wise intensities: $|\mathbf{F}_+(\mathbf{r},|\mathbf{k}|)|^2$ on the left, and $|\mathbf{F}_-(\mathbf{r},|\mathbf{k}|)|^2$ on the right. The monochromatic $\mathbf{F}_\lambda(\mathbf{r},|\mathbf{k}|)$ are obtained as in Eq. 2, but using single frequency complex electric and magnetic fields. To a good approximation, and in contrast with what Fig. 4b) shows, Fig. 4a) shows that the disk does not couple the two helicities upon light-matter interaction: It preserves the incident helicity. In its most general illumination-independent embodiment, helicity preservation is achieved by objects exhibiting electromagnetic duality symmetry, a fundamental continuous symmetry in electromagnetism, whose action is:

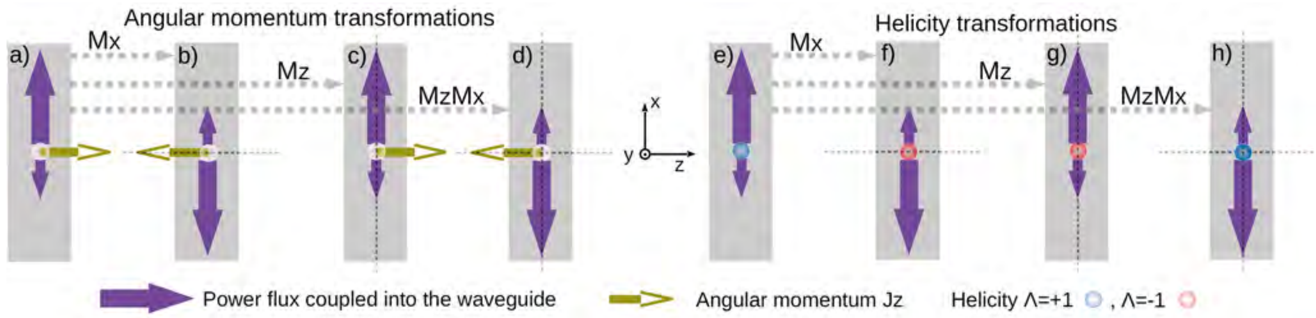
$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &\rightarrow \mathbf{E}^\theta(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\cos\theta - c_0\mathbf{B}(\mathbf{r},t)\sin\theta, \\ c_0\mathbf{B}(\mathbf{r},t) &\rightarrow c_0\mathbf{B}^\theta(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\sin\theta + c_0\mathbf{B}(\mathbf{r},t)\cos\theta, \end{aligned} \tag{4}$$

for a real angle θ, and its action on the helical fields

$$\mathbf{F}_\lambda(\mathbf{r},t) \rightarrow \mathbf{F}_\lambda^\theta(\mathbf{r},t) = \mathbf{F}_\lambda(\mathbf{r},t)\ \exp(-\lambda i\theta) \tag{5}$$

reveals that, in the same way that rotations are generated by angular momentum operators, *e.g.* $R_z(\theta) = \exp(-i\theta J_z)$, duality is generated by the helicity operator $D_\theta = \exp(-i\theta\Lambda)$. The polarization of the field can then be treated within the powerful framework of symmetries and conservation laws. Such

Figure 3. Emitters (circles) on top of waveguides (gray boxes) emit light with a definite angular momentum along the z-axis (yellow-brown arrows), and a definite helicity (blue/red circles), which couples asymmetrically to the waveguide modes, with more power going towards the +x direction than to the -x direction, or vice versa (purple arrows). The initial situations [panels a) and e)] are transformed by mirror reflections which are symmetries of the waveguide and leave the position of the emitter unchanged: Mx [panels b) and f)], Mz [panels c) and g)], and the composition MxMz [panels d) and h)], respectively, where *α* indicates a reflection across the plane perpendicular to the *α* direction. In each panel, the origin of coordinates is at the position of the emitter, and the coordinate axes are oriented as shown in the central part of the figure. The transformation properties of helicity allow us to conclude that it cannot be responsible for the directional coupling: For example, e) and h) show the same helicity producing the opposite directionality. Such contradictions do not arise for angular momentum, which is indeed the property of light which controls directionality [5].



framework facilitates the identification of design guidelines for achieving specific effects, such as zero back-scattering (anti-reflection) (see Chap. 4 in [4]), and enhanced near-field interaction with chiral molecules (see Chap.5.3 in [4], [7]). Helicity preservation is one of the guidelines in these two particular cases. Unfortunately, dual-symmetric systems are hard to fabricate. The theoretically most straightforward way to design a dual system is by using materials with equal relative electric permittivity and magnetic permeability: a very demanding requirement. Fortunately, the essential idea in duality symmetry, balanced electric and magnetic responses, allows to design non-magnetic systems that preserve helicity, at least for particular frequencies and illumination directions. For example, the aspect ratio of the silicon disk in Fig. 4 was optimized [7] for hosting equal electric and magnetic dipole moments upon on-axis illumination at f = 125 THz.

A SCALAR PRODUCT PROVIDES MORE TOOLS

The appropriate scalar product for Maxwell fields is [1]:

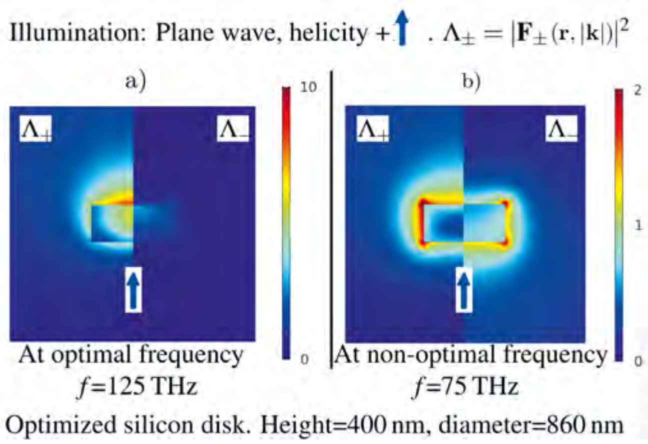
$$\langle F|G\rangle = \int_{\mathbb{R}^3 - \{0\}} \frac{d\mathbf{k}}{\hbar c_0|\mathbf{k}|}\left[\mathbf{F}_+(\mathbf{k})^\dagger \mathbf{G}_+(\mathbf{k}) + \mathbf{F}_-(\mathbf{k})^\dagger \mathbf{G}_-(\mathbf{k})\right] \qquad (6)$$

where $|F\rangle$ and $|G\rangle$ are two different sets of Maxwell fields, specified by their corresponding $\mathbf{F}_\lambda(\mathbf{k})$, and $\mathbf{G}_\lambda(\mathbf{k})$. Equation 6 is conformally invariant: Its numerical value is identical to the scalar product between $X|F\rangle$ and $X|G\rangle$, for any transformation X in the conformal group, the 15 parameter group which is the largest group of invariance of Maxwell equations [8].

Equation 6 is defined for free Maxwell fields which do not interact with matter. Nevertheless, in a light-matter interaction sequence such as the one depicted in Fig.5a), Eq. 6 can be applied to the pre- and post-interaction fields, only excluding the grayed out period where the interaction is ongoing.

The scalar product for Maxwell fields allows to use the tools of Hilbert spaces in electromagnetism, and to leverage ● ● ●

Figure 4. Helical decomposition of the scattered field around a silicon disk illuminated on axis with two different frequencies (adapted from [7]).



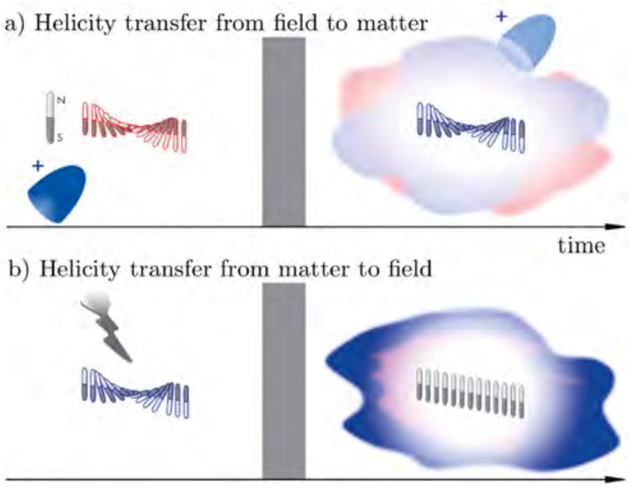


Figure 5. Helicity exchange between electromagnetic fields and material magnetization. In a) a beam of positive helicity interacts with a magnetization of the opposite helicity and flips its handedness. In b), an initially chiral magnetization is turned into an achiral one by achiral and non-necessarily optical forces. In this process, the material radiates an electromagnetic field which contains (part of) the helicity lost by the material.

some of the methods from quantum mechanics. Given an electromagnetic field $|F\rangle$, the average of its linear momenta, angular momenta, energy, and any other quantity represented by a hermitian operator Γ is $\langle F|\Gamma|F\rangle$, which can be readily written down as a k-space integral [2] using Eq.6.

Recently, the k-space integral expression of the average electromagnetic helicity, $\langle F|\Lambda|F\rangle$, has lead to the definition of material helicity [3]. Matter in static equilibrium is electromagnetically represented by its electric charge density $\rho(\mathrm{r})$, and its magnetization density $\mathbf{M}(\mathrm{r})$, which are bijectively connected to the static fields that they generate at $\omega = 0$. It is then possible to define the total helicity, $\langle\Lambda\rangle$, as the sum of two terms: The electromagnetic helicity, $\langle F|\Lambda|F\rangle = \langle\Lambda_{\omega>0}\rangle$ that measures the difference between the number of left-handed and right-handed photons of the free field, and a static term, $\langle\Lambda_{\omega=0}\rangle$, that measures the screwiness of the static magnetization density in matter:

$$\langle\Lambda\rangle = \langle\Lambda_{\omega>0}\rangle + \langle\Lambda_{\omega=0}\rangle, \text{ where}$$

$$\langle\Lambda_{\omega>0}\rangle = \int_{\mathbb{R}^3 - \{0\}} \frac{d\mathbf{k}}{\hbar c_0|\mathbf{k}|} |\mathbf{F}_{+}(c_0|\mathbf{k}|, \mathbf{k})|^2 - |\mathbf{F}_{-}(c_0|\mathbf{k}|, \mathbf{k})|^2,$$

$$\langle\Lambda_{\omega=0}\rangle = \int_{\mathbb{R}^3 - \{0\}} \frac{d\mathbf{k}}{\hbar c_0|\mathbf{k}|} \frac{|\mathbf{M}_{+}(0, \mathbf{k})|^2 - |\mathbf{M}_{-}(0, \mathbf{k})|^2}{2/\mu_0} \qquad (6)$$

The $\mathbf{F}_{\pm}(c_0|\mathbf{k}|, \mathbf{k})$ are the $\mathbf{F}_{\lambda}(\mathbf{k})$ of Eq.3, and $\mathbf{M}_{\lambda}(0, \mathbf{k})$ are the $\lambda = \pm 1$ helicity components of the 3D Fourier transform $\mathbf{M}(0, \mathbf{k})$ of the static magnetization density $\mathbf{M}(\mathbf{r})$: $2\mathbf{M}_{\lambda}(0,\mathbf{k}) = (I + \lambda i\hat{\mathbf{k}}\times)(i\hat{\mathbf{k}}\times)^2\mathbf{M}(0,\mathbf{k})$. The charge density $\rho(\mathrm{r})$ does not host any material helicity because the electric field that it produces is longitudinal (parallel to $\mathbf{k}$), and is hence annihilated by the action of the helicity operator $\Lambda \equiv i\hat{\mathbf{k}}\times$. Importantly, $\langle\Lambda_{\omega=0}\rangle$ can be readily shown to be $1/2\sqrt{\epsilon_0/\mu_0}$ times the magnetic helicity. The electromagnetic helicity of the dynamic Maxwell fields and the static magnetic helicity [9] are unified as two different embodiments of the same physical quantity, establishing the theoretical basis for studying the conversion between the two embodiments of total helicity upon light-matter interaction [3], as illustrated in Fig.5.

OUTLOOK

The design of helicity preserving systems is an area of research with technological implications. While approximations such as the geometric optimization illustrated in Fig.4 can already be of some use, recent advances in molecular science prompt the question of whether a truly dual symmetric material, albeit likely with a narrow operational frequency band, can be achieved through molecular design.

Finally, the transfer of helicity from the field into a magnetic material is a mechanism for affecting the chirality of matter in static equilibrium, that is, in its ground state. This points to possible applications in magnetization control by optical means. The potential role of the effect in the all-optical helicity-dependent magnetization switching [10] is an exciting research question.

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