LIGHT AND COLOR: PHOTONIC RESOURCES FOR QUANTUM METROLOGY

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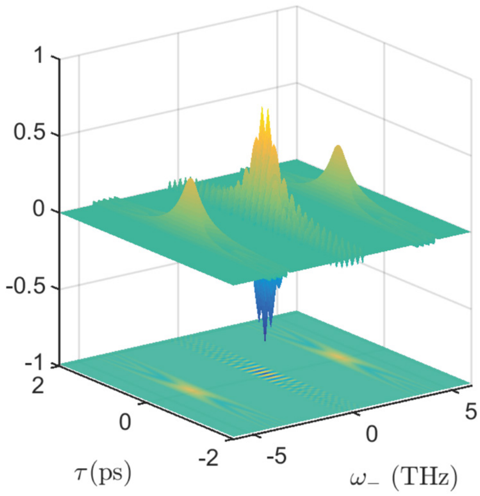

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A finite precision is always associated with the estimation of a parameter, whether it is time, length, or a rotation angle. What does fundamentally limit this precision, given a fixed amount of resources—such as total energy—that are used in parameter estimation? The answer depends on whether classical or quantum strategies are employed. In this paper, we explore the fundamental precision limits of metrological protocols, with a focus on how quantum optics and photonic devices can achieve a quadratic improvement in precision compared to classical setups while using the same amount of resources. We analyze both theoretical aspects and experimental demonstrations, highlighting the respective roles of field statistics and spectral properties in enhancing precision.

BASIC PRINCIPLES OF QUANTUM METROLOGY

Precision plays an essential role, not only in technology, but also in our daily life. Optimizing precision is particularly important when it relates to time measurements: time is considered by some as our most precious resource - or the most important parameter. In optical setups, the precision associated with time (or temporal delay) measurements, is often related to the field's spectral properties. For instance, for classical monochromatic fields, the time estimation precision limit is roughly said to be the inverse of the field's frequency. However, in quantum optics, light is characterized not only by its modal properties - such as spatial or frequency modes - but also by the statistical distribution of photons. This dual nature encapsulates the interplay between the wave-like and particle-like aspects of light. The study of parameter estimation and the associated precision is called metrology. In this contribution, we will discuss how quantum optical strategies can be applied to this field.

We start by recalling the basic principles of metrology, and its quantum counterpart. We will focus throughout this contribution into time estimation precision, but the description provided is valid for other parameters as well. The steps of a parameter estimation protocol are outlined in the pink squares of Box 1, that we will further explore in the context of quantum optics. The first step of the protocol involves generating a quantum state, such as a squeezed state or an entangled multi-photon state—both of which will be examined in greater details later in this article. This initial state serves as a probe, incorporating both modal and statistical properties.

In the "Evolution" step, information about the parameter to be measured (denoted τ) is imprinted onto the state. For instance, in the case of time estimation, this can occur through the free evolution of the field, which depends on the parameter (time) and is governed by a Hamiltonian (specifically, the free-field Hamiltonian). Next, a measurement is performed, yielding an outcome x with probability $P_\tau(x)$, which depends on the parameter to be estimated τ. From these outcomes, we define an estimator—a function that maps measurement results to the estimated parameter. The average value of this estimator over many experimental runs provides an estimate of the parameter's value, with a certain precision. Naturally, the higher the precision, the more accurate the measurement.

The precision limit can be calculated using the Fisher information, defined as $\mathcal{F}=\sum_i^M \frac{1}{P_\tau(x_i)}\left(\frac{\partial P_\tau(x_i)}{\partial \tau}\right)^2$, where x_i are the possible measurement outcomes of the experiment. Given a state and a parameter, there are a number of different measurements that can be implemented to infer the value of this parameter, leading to different probabilities $P_\tau(x_i)$. Using $\mathcal{F}$, we see that the precision associated with the measurement of the parameter τ is limited by $\delta\tau \geq \frac{1}{\sqrt{N\mathcal{F}}}$, where N is the number of repetitions of the experiment. This precision limit is known as the Cramér-Rao bound.

Increasing the precision of a given parameter estimation protocol involves finding measurement strategies such that $\mathcal{F}$ is as large as possible. But how large can $\mathcal{F}$ be? By optimizing the precision over all possible measurement strategies, we can define the quantum Fisher information (QFI): $\mathcal{Q}=max\,\mathcal{F}$, leading to the quantum precision limit, $\Delta\tau \geq \frac{1}{\sqrt{N\mathcal{Q}}}$, called the quantum Cramér-Rao bound. Hence, the best possible precision occurs when $\mathcal{F}=\mathcal{Q}$. And what limits the QFI ? For pure states undergoing Hamiltonian evolution (see Box 1), the QFI takes a relatively simple form: $\mathcal{Q}=4\Delta^2\hat{H}$, where $\hat{H}$ is the Hamiltonian that generates the parameter dependent evolution into the state considered as a probe. Hence, given a Hamiltonian, the maximal achievable precision depends entirely on the choice of probe state. This is where quantum resources may offer advantages over classical ones.

Indeed, if we consider a system composed of K qubits, we have that each qubit can be prepared not only in states $|0\rangle$ and $|1\rangle$, the analogous of the classical bits 0 and 1, but also in states as $|\Psi\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, which

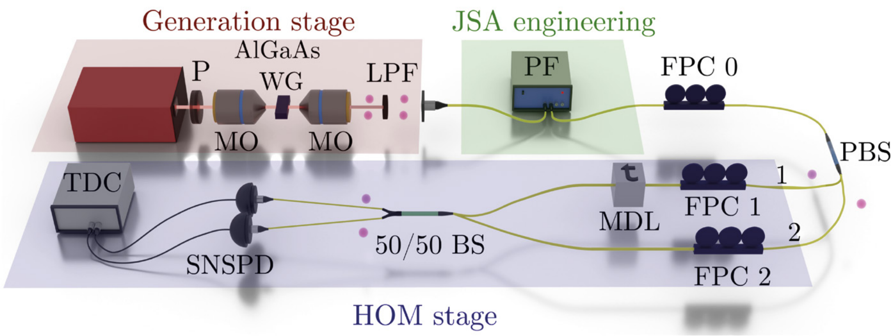


Figure 1. Experimental setup for investigating the metrological performance of the Hong-Ou-Mandel (HOM) experiment, showing the generation, joint spectral amplitude (JSA) engineering, and HOM interferometer stages.

is a quantum superposition of the values 0 and 1 with no classical analog. By combining $K > 1$ qubits, more complex states can be created. We can either produce separable states, as $|\Psi_1\rangle_1|\Psi_2\rangle_2...|\Psi_K\rangle_K$, where the qubits are independent from one another, or entangled states, as $\frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2...|0\rangle_K+|1\rangle_1|1\rangle_2...|1\rangle_K)$. This state, analogous to a Schrödinger cat states, has no classical counterpart. While for separable states it has been shown that the QFI scales as $\mathcal{Q}\leq K$, for entangled states we have that $\mathcal{Q}\leq K^2$, so such states can lead up to a quadratic enhancement of precision. This is also the maximal possible precision enhancement that can occur in quantum metrology, and it is called the Heisenberg limit. Importantly, this enhancement occurs without increasing the resources which are, in the present case, the number of qubits. We now see how these results manifest in the context of quantum optics.

USING QUANTUM OPTICS FOR QUANTUM METROLOGY

We now detail the specific case of time precision limits in quantum optics. The free evolution leads to a time dependent phase to photon states, and temporal delays can be easily implemented in interferometers by creating a path length difference between two propagation modes. The recombination of the propagation modes leads to interference, which enables the measurement of this delay. The QFI associated with the estimation of temporal delays can then be calculated from the free propagation Hamiltonian and the state used as a probe. For a single frequency mode field, the free propagation Hamiltonian is given by $\hbar\omega\hat{n}$ where ω is the field's frequency and $\hat{n}$ an operator counting the number of photons of the field. However, it takes a slightly more complex form in the general case, when the field can be decomposed into different frequency modes. In this case, we can still obtain an intuitive expression for the QFI, which reads, for a single mode state, $\mathcal{Q} = 4(\Delta^2\omega\bar{n}+ \Delta^2\hat{n}\bar{\omega}^2)$, where $\Delta^2\omega$ is the frequency variance of the mode, $\bar{\omega}$ is average frequency, $\bar{n}$ is the average number of photons, and $\Delta^2\hat{n}$ the photon number variance. In general, the total photon number is not fixed, and paradigmatic states, as Gaussian states (squeezed and coherent states), consist of superpositions of different photon numbers. The expression of the QFI for a single mode is elegant, for treating symmetrically modal (frequency variables) and statistical (photon number variance) properties of the field.

In early quantum optics metrology, only single-mode states were considered. Coherent states, which are classical-like states, are an example of a single-mode state. They are a special case of Gaussian states for which the photon number follows a Poissonian distribution. The QFI of a coherent state is given by $\mathcal{Q} = 4\bar{n}\bar{\omega}^2$. We see that the average photon number appears here as the analog to the number of qubits K, and we recover the known classical result that the precision in time estimation is limited by the inverse of the frequency for monochromatic fields. Photons in a coherent state behave as independent particles, explaining the observed analogy with the separable states of qubit systems described earlier.

If we fix the average field's energy as the available resource (i.e., $\hbar\bar{n}\bar{\omega}$), how can we obtain the quadratic scaling in precision observed in qubit-based quantum metrology? It has been shown [1] that the optimal metrological strategy concentrates all resources into a single mode. In this case, the frequency variance appears as a classical-like quantity, and doesn't play a role in the scaling of precision enhancement. However, some non-classical states can enable $\Delta^2\hat{n}=\bar{n}^2$ Again, the average photon number plays the same role as the number of qubits, leading to a quadratic precision enhancement—the Heisenberg limit.

The previous results assume that modal properties (frequency) and statistical properties (photon number) are independent. However, this is not always the case. A key example is photon pairs, commonly used in Bell inequality experiments [2]. These states exhibit mode and particle entanglement and cannot be described as single-mode states. Frequency modes can also be entangled for photon pairs [3], and in this case, the QFI must be modified to account for mode correlations.

To explore this scenario, let us consider a system with K photons and K propagation modes. Each mode can host a single photon with its own frequency distribution. The free evolution Hamiltonian now depends on both propagation direction and frequency distribution in each mode. For a single propagation mode, the QFI follows from the single-mode expression: $\mathcal{Q}_i=4\Delta^2\hat{\omega}_i$ (where we have used the general expression for the QFI of a single mode, labled i, in the case where $\bar{n}_i = 1$, one photon per propagation mode). For K independent photons (in independent propagation modes), if we suppose, for simplifying reasons and without loss of generality, that all the variances are the same, the total QFI is given by $\mathcal{Q}=\sum_{i}^{K}\mathcal{Q}_i=4\Delta^2\hat{\omega}\,K$. This result resembles the QFI of coherent states and separable qubit states, since the precision scales linearly with the number of photons.

We can now consider a system of maximally entangled photons, both in frequency and in propagation direction. An example of such a state is $\frac{1}{\sqrt{2}}(|\omega_1\rangle\ldots|\omega_K\rangle+|\omega'_1\rangle\ldots|\omega'_K\rangle)$, where $\omega_i\neq\omega'_i$ for all values of i, that labels auxiliary modes. This state is the generalization of a pair of polarization entangled photons, where each photon is on a different propagation mode. For this state, the frequency variance in each mode is $\Delta^2\omega_i=\frac{1}{4}(\omega_i-\omega'_i)^2$. If the variance is the same for each propagation direction, the QFI becomes $\mathcal{Q}=\Delta^2\omega K^2$. Then, analogously to the case of entangled qubits, we recovered the quadratic scaling of precision with the number of particles (here, photons instead of qubits). However, now precision is also proportional to the frequency variance for each mode. Interestingly, in this case, we observe at the same time a wavelike behavior of the scaling and a particle-like one.

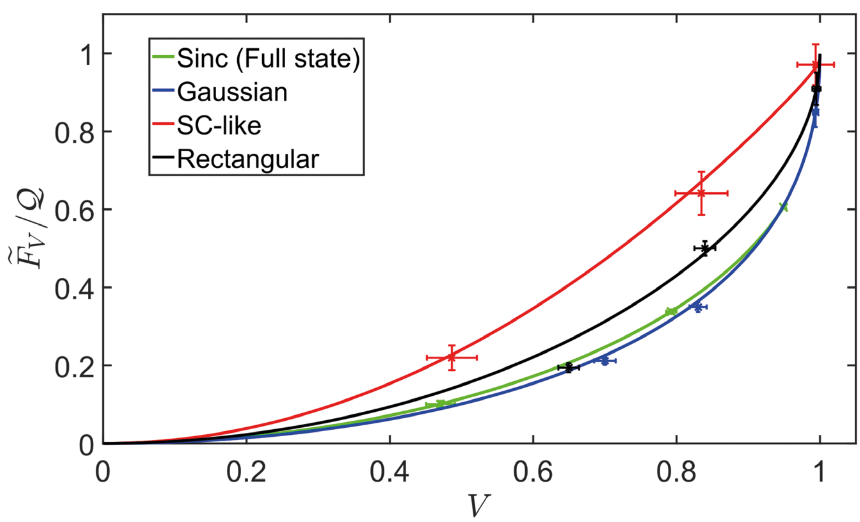
We have provided a broad picture highlighting how both modal correlations and particle statistics can lead to precision enhancement in quantum optics [4]. Nevertheless, two natural questions appear: the first one is, how can one engineer frequency entangled states, that permit achieving the Heisenberg precision limit ? On the other hand, what is the optimal measurement strategy, for which the Fisher information is equal to the QFI for this type of system ? We discuss these two questions in the next section.

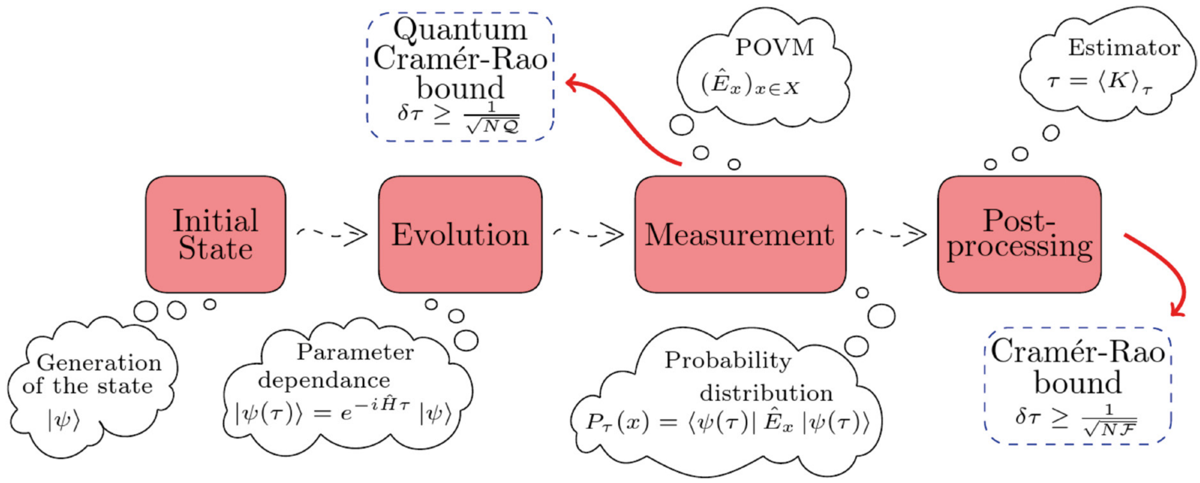
QUANTUM METROLOGY USING THE HONG-OU-MANDEL EXPERIMENT

Theoretically, the Hong-Ou-Mandel (HOM) interferometer [5] was shown to be an optimal measurement tool as it allows the quantum Cramér-Rao bound to be reached under ideal conditions of perfect visibility, where photons either perfectly bunch or anti-bunch. Not only it enables reaching the QFI, but also, in principle, the Heisenberg scaling of precision (even though this type of experiment is limited to two photons, precision is proportional to the square of the number of photons). Several experiments have shown the approach to this optimal precision using different two-photon frequency entangled states [6].

However, in reality, perfect visibility is unattainable due to experimental limitations. Interestingly, the •••

Figure 2. Scaling of the ratio $\tilde{F}_V/\mathcal{Q}$ for different biphoton states with respect to the HOM visibility V. For SC-like states a maximal value of $\tilde{F}_V/\mathcal{Q}$ is attained for $V = 99.4\%$.





Basic principles (pink boxes) of a metrological protocol, using classical or quantum resources. The boxes and thought bubbles correspond to relevant actions of quantities which are specific to quantum protocols. The "generation of the state" is considered as a black-box outputting a given quantum optical state (as an entangled or squeezed state). In "Evolution", a unitary operator generates a parameter depending evolution, and the resulting state now depends on the parameter to be estimated (τ in the example). The "Measurement" step can also be formalized as the application of Positive Operator Valued Measurements (POVMs), and the probability of obtaining an outcome x depends on the parameter τ. In the post-processing, this probability can be used to infer, by using an estimator K, the value of the parameter κ from the probability distribution $P_\tau(x)$ obtained from the measurements. The precision $\delta\tau$ associated with this procedure on the obtained value of τ can be computed using the Cramér-Rao bound, where N is the number of repetitions of the experiment and $\mathcal{F}$ is the Fisher information (details in the main text). The Fisher information depends on the measurement performed (the POVM $\hat{E}_x$). An optimization over all possible measurements leads to the quantum Fisher information $\mathcal{Q}$ and to the quantum Cramér-Rao bound, which can be up to quadratically smaller than the Cramér-Rao bound (depending on the chosen initial state), leading to a better-than-classical precision.

consequences of this fact are extreme: a finite visibility leads to the impossibility of reaching the QFI under any circumstances in this experiment. It is however possible to understand the approach to the maximal possible Fisher information in this set-up: the effects of a finite visibility highly depend on the spectrum of the entangled two-photon state (and not on its variance, as previously). In order to better understand these points, we recall the basic principles of the HOM experiment and interpret it as a metrological protocol.

In the HOM setup, two photons in two different propagation modes (as in the previous model) are injected into the two input ports of a 50/50 beam splitter. One of the modes is temporally delayed with respect to the other: this delay is the parameter to be measured, and the free propagation Hamiltonian is the time (parameter) dependent generator of the evolution. After the beam-splitter, either both photons exit through the same port (bunching effect), or they exit through opposite ports (anti-bunching effect). Coincidence detection at the outputs determines whether the photons have bunched or not. The time delay in one path changes the distinguishability of the photonic modes, altering the coincidence detection probability and enabling measurements of time or path differences. The precision we compute is associated with this measurement, and the idea is to optimize the Fisher information associated with this quantity.

We have demonstrated in [7] how the spectral properties of the quantum state affect the scaling of precision with visibility. To explore this, we engineered various types of frequency entangled two-photon states and tuned the visibility of the HOM interference. The quantum source employed is an AlGaAs Bragg reflection waveguide, which generates polarization-entangled photon pairs *via* type-II spontaneous parametric down-conversion. As illustrated in Fig.1, a continuous-wave laser with a wavelength of 772.42 nm pumps the waveguide, producing pairs of horizontally and vertically polarized photons that are collected in single-mode fibers and directed to a programmable filter. This filter enables the engineering of the spectral distribution (joint spectral amplitude) of the photons. This step corresponds to the "Initial state" preparation in Box 1. At the filter output, the photon pairs are separated - the "Evolution" step in Box 1 - with horizontally and vertically polarized photons entering the HOM interferometer through arms 1 and 2, respectively. Precise control over polarization distinguishability—and consequently, HOM visibility—is achieved using two polarization controllers (FPC1 and

FPC2), which allow for arbitrary polarization transformations. The temporal delay between the photons is controlled using an optical delay line (MDL) placed in arm 1. The two paths are recombined and separated by a 50/50 beam-splitter (BS). Finally, the photons are sent to superconducting nanowire single photon detectors (SNSPD) - the "measurement" step in Box 1. Temporal correlations between the detected photons are analyzed by a time-to-digital converter (TDC).

The experimental results display an excellent agreement with the theoretical predictions: a quadratic behavior is observed for the ratio between the maximal Fisher information $\tilde{F}_V$ and the QFI, $\tilde{F}_V/\mathcal{Q}$. The curvature of the parabola depends on the state, and we confirm experimentally that for spectral distributions corresponding to Schrödinger cat-like states (states of the type $\frac{1}{\sqrt{2}}(|\omega_1\rangle_1|\omega_2\rangle_2+|\omega_2\rangle_1|\omega_1\rangle_2)$, precision decreases slower with visibility than for states with a Gaussian spectral distribution is $\int d\omega p(\omega)|\omega\rangle_1|\omega+\delta\rangle_2$, where is a Gaussian distribution and is a constant frequency. The experimental results are shown in Figure 2.

Figure 2 illustrates the evolution of the ratio between the maximum of the Fisher Information $\tilde{F}_V$ and the quantum Fisher Information $\mathcal{Q}$ as a function of visibility V for four engineered states. The results reveal that the Schrödinger Cat (SC)-like state exhibits the most favorable scaling behavior, outperforming the Gaussian state. For instance, at a visibility of approximately 99.4%, the $\tilde{F}_V/\mathcal{Q}$ ratio is 0.97 for the SC-like state and 0.85 for the Gaussian state. Notably, the experimentally achieved ratio of 0.97 relative to the quantum limit is the highest ever reported, establishing a new benchmark in the field.

CONCLUSION

In conclusion, we have presented the fundamental ideas and concepts of quantum metrology in quantum optics, along with recent efforts to establish a unified framework for quantum metrology ranging from single photons in different modes to many photons in a single mode. This approach highlights the respective roles of field statistics and of modes of the field, which vary depending on the specific quantum state. Additionally, we have described an experiment that explores the metrological potential of a two-photon system, along with its limitations. A careful analysis of these limitations has led to a strategy for optimizing the ratio between the Fisher information and the quantum Fisher information in this type of experiment. Quantum metrology in quantum optics remains a rapidly evolving field. A more comprehensive understanding of the interplay between modes and states will be essential for fully harnessing the capabilities of quantum optics to reach ultimate precision limits.

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